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OPTIMAL ESTIMATION POLICIES OF STOCHASTIC LINEAR SYSTEMS WITH TIME-VARYING PARAMETERS

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Abstract—The paper is concerned with the problem of designing optimal strategies for precise parameter estimation in the context of regression models with stochastically varying coefficients. The maximum accuracy problem considered in this paper can also be treated as an initial phase of a stochastic control problem. Before the control policies are implemented, an estimation phase is introduced to determine the unknown model parameters. This will avoid solution to a difficult dual control problem.

1. INTRODUCTION

Experimentation has always been an essential ingredient of research for physical scientists and psychologists, educators and sociologists, since testing of most hypotheses requires the capability of conducting controlled and repeatable experiments. Such methods, however, have only recently been applied in economics and business.[1]

A properly designed and executed social experiment can provide the strongest evidence that a certain policy action actually causes or, if implemented, would cause a given result. The great cost, size and administrative complexity of such experiments, however, make them different from the classical experiments in physical sciences, agriculture or psychology which were developed for simpler situations.

The growing interest in controlled social experimentation has led economists to pay more attention to the appropriate design of such experiments. When the model under consideration is a classical static regression model, the analysis of experimental design is straightforward and has been discussed in the literature (see [3,38]). If, however, the model is dynamic with stochastically

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[1] It seems a little time ago that economists viewed experimentation as a tool available to physical scientists and psychologists, educators and sociologists, but not to them. Economists are beginning to see, however, that experiences generated from simple controlled settings can be used as criteria for determining the relative acceptability of general theories and related models of complex economic systems. Although there still are economists who think that experimental methods are in principle not applicable in economics, controlled experimentation in economics is becoming more and more common and scientific thinking is shifting to a qualified acknowledgment that the experimental methods are applicable when the economic problems are carefully defined.

Several large-scale experiments in negative income taxation have been conducted, including the New Jersey-Pennsylvania [1–6] and [7,8], Rural [9,10]; Gary [11] and Seattle and Denver Income Maintenance Experiments [12,13].

Other examples of real-world experiments, attempting to measure responsiveness to various types of economic incentive programs, are (1) the housing demand experiment, which was designed to find out how household expenditures for housing were related to various forms and levels of housing allowance [14]; (2) the health insurance experiments, whose concern centered on finding out how individual use of medical care relates to the coinsurance and deductible features of health insurance policies [15] and (3) the peak-load pricing of electricity experiments, whose principal goal was the measurement of residential customer responsiveness to charging higher rates during hours and seasons of higher demand [16–19]. In the environmental area, the experimental method is being explored as a tool to elicit individual preferences about environmental variables [20] and experiments on the effectiveness of various pollution taxes could now be laying the foundation of new antipollution laws [21]. Finally, game experiments [22–33] and computer simulation experiments [34–37] are other economic contexts in which experimentation is applicable.

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varying coefficients and if time series data are to be collected then the analysis becomes much more complex.[2]

The purpose of this paper is to use control theoretic concepts to develop a methodology for designing optimal strategies for precise parameter estimation in the context of regression models with time-varying coefficients. It must be pointed out, however, that, even though no control aspect is included in the analysis, the experimental design problem can be treated as an initial phase of a stochastic control problem of a dynamic system. In such a situation, it is desirable that the parameters be determined as quickly as possible. This phase can be used to estimate the parameters up to a desirable accuracy and complete learning will eventually take place during the control phase. Although this separation of estimation and control differs from the definition of dual control where estimation and control take place simultaneously,[3] it provides an alternative approach to indirect adaptive control and thus avoids the solution of a difficult dual control problem.

2. PROBLEM FORMULATION

Consider the following model

$$z_t = \mathbf{u}_t \underline{\gamma}_t + e_t, \tag{1}$$

for $t = 1, 2, \ldots, T$, where $\mathbf{u}_t = (u_{1t} u_{2t} \ldots u_{mt})$ and $\underline{\gamma}_t = (\gamma_{1t}\gamma_{2t}\ldots\gamma_{mt})'$. Here the z_t are scalar observations on the dependent variables; $\{\gamma_{it}\}_{i=1}^{m}$ are scalar parameters for time t and the $\{u_{it}\}_{i=1}^{m}$ are observed values of the exogenous (control) variables at time t. Finally, the e_t are white noise disturbances with common (known) variance σ^2.

We now assume that $\underline{\gamma}_t$ varies according to

$$\underline{\gamma}_t = \underset{\sim}{\Gamma}_t \underline{\gamma}_{t-1} + \mathbf{v}_t, \tag{2}$$

$$\left\{\mathbf{v}_t \sim N\left(\mathbf{O}; \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right); \quad \underline{\gamma}_{\circ} \sim N\left(\hat{\gamma}_0; \hat{\Sigma}_{\underline{\gamma}_{\circ}}\right)\right\}, \tag{3}$$

where $\left\{\underset{\sim}{\Gamma}_t; \underset{\sim}{\Sigma}_{\mathbf{v}_t}; \hat{\underline{\gamma}}_{\circ}; \hat{\Sigma}_{\underline{\gamma}_{\circ}}\right\}$ are assumed known and the sequences e_t and $\mathbf{v}_t$ are assumed uncorrelated.

For any given values of the model parameters, $\left\{\underset{\sim}{\Gamma}_t; \underset{\sim}{\Sigma}_{\mathbf{v}_t}; \hat{\underline{\gamma}}_{\circ}; \hat{\Sigma}_{\underline{\gamma}_{\circ}}, \sigma^2\right\}$, model (1) and the "state" equation system (2)–(3) are in the standard discrete dynamic linear form and the Kalman filter [45,46] can be constructed. The Kalman filter algorithm will generate the estimate $\hat{\underline{\gamma}}_t$ of the "state" $\underline{\gamma}_t$ and the variance-covariance matrix of the error vector $\underline{\epsilon}_t = (\underline{\gamma}_t - \hat{\underline{\gamma}}_t)$, respectively. The estimate $\hat{\underline{\gamma}}_t$ is the minimum variance (and unbiased under appropriate assumptions) estimate of $\underline{\gamma}_t$. Hence,

$$\begin{aligned}\hat{\underline{\gamma}}_t = \Big(\underset{\sim}{\Gamma}_t \hat{\underline{\gamma}}_{t-1} &+ \left[\underset{\sim}{\Gamma}_t \mathbf{V}\left(\hat{\underline{\gamma}}_{t-1}\right)\underset{\sim}{\Gamma}_t' + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right] \\ &\times \mathbf{u}_t'\left[\sigma^2 + \mathbf{u}_t\left(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right]\underset{\sim}{\Gamma}_t' + \Sigma_{\mathbf{v}_t}\right)\mathbf{u}_t'\right]^{-1}\left[z_t - \mathbf{u}_t \underset{\sim}{\Gamma}_t \hat{\underline{\gamma}}_{t-1}\right]\Big),\end{aligned} \tag{4a}$$

$$\underline{\gamma}_{\circ} = \hat{\underline{\gamma}}_{\circ}. \tag{4b}$$

$\mathbf{V}(\hat{\underline{\gamma}}_t)$ is the state error covariance, that is,

$$\mathbf{V}(\hat{\underline{\gamma}}_t) = E\left\{\left[\hat{\underline{\gamma}}_t - \underline{\gamma}_t\right]\left[\hat{\underline{\gamma}}_t - \underline{\gamma}_t\right]' \mid z_{t-1}, z_{t-2}, \ldots z_0\right\}$$

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[2] Econometricians have paid increasing attention in recent years to regression models in which the coefficients are thought to evolve over time, following a stochastic process that can be either stationary or nonstationary (see, for example, [39] and [40]).

[3] See, for example, [41–44].

and satisfies the following Ricatti difference equation:

$$\begin{aligned}\mathbf{V}(\hat{\underline{\gamma}}_t) = &\left(\underset{\sim}{\Gamma}_t\mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right]\underset{\sim}{\Gamma}_t' + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right) - \left(\underset{\sim}{\Gamma}_t\mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right]\underset{\sim}{\Gamma}_t' + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right) \\ &\times \mathbf{u}_t\left(\sigma^2 + \mathbf{u}_t\left[\underset{\sim}{\Gamma}_t\mathbf{V}\left(\hat{\underline{\gamma}}_{t-1}\right)\underset{\sim}{\Gamma}_t' + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right]\mathbf{u}_t'\right)^{-1}\mathbf{u}_t\left(\underset{\sim}{\Gamma}_t\mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right]\underset{\sim}{\Gamma}_t' + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right), \end{aligned} \tag{5a}$$

$$\underset{\sim}{\Sigma}_{\underline{\gamma}_o} = \hat{\underset{\sim}{\Sigma}}_{\underline{\gamma}_o}. \tag{5b}$$

Now suppose that attention is limited to n design points which have been chosen by the experimenter (or the sponsor of the experiment) so that the relevant region of the design space is adequately covered (see, for example, [38,47]). Limiting our attention to n admissible design points may be viewed as a two part assumption: first, that observations must be restricted to a given region in the design space (a matter of necessity); and second, that within that region they must fall at only n points (a matter of convenience). It should be clear, however, that actual determination of the number of design points and their exact specification is not a trivial task and depends on many factors, including a prior knowledge of the appropriate range of variation for policy purposes of each design variable.

In addition, it is assumed that, at each period t, the researcher is able to choose only one out of the n admissible design points. Mathematically, we can express this by the following constraints

$$\left(\ell_{jt} = \begin{cases} 1, & \text{if the } j^{\text{th}} \text{ design point is chosen at time } t \\ 0, & \text{otherwise} \end{cases} ; \sum_{j=1}^{n} \ell_{jt} = 1\right). \tag{6}$$

Finally, if different design points have different costs and the investigator has a budget constraint (the typical case in economics), the experimenter (or the sponsor of the experiment) has to specify the unit cost of an observation at the j^{th} design point and the total budget available for all $t = 1, 2, \ldots, T$. More precisely, the budget constraint can take the following form:

$$\sum_{t=1}^{T}\sum_{j=1}^{n} \ell_{jt}c_{jt} = \Lambda_T; \qquad c_{jt} > 0 \quad \text{for all } j = 1, \ldots, n;\ t = 1, 2, \ldots, T, \tag{7}$$

where c_{jt} is the j^{th} design cost at time t and Λ_T is the total budget.[4]

As different scientists may have different definitions of information, one can find in the literature more than one concept and measure of information.[5] For the purpose of this paper, information provided by a sample can be measured by the (Fisher) information matrix $\mathbf{I}$, defined as the negative of the Hessian of the log-likelihood function with respect to the unknown parameters. If the parameters are estimated by a best linear unbiased estimator, then $\mathbf{I}^{-1}$ is proportional to the variance-covariance matrix of the estimator. This suggets that the amount of information is closely related to the precision of the parameter estimates.

It is clear that given the initial condition $\hat{\underset{\sim}{\Sigma}}_{\underline{\gamma}_o}$ and given any design sequence, which satisfies the constraints (6) and (7), we can obtain by straightforward calculation the values of the matrix $\mathbf{V}(\hat{\underline{\gamma}}_t)$ for all t. Of course, the values of the elements of this matrix will depend on the design sequence which was chosen. In this context, the researcher may view Equations (5) as defining a dynamical system, whose state variables are the elements of the matrix $V(\hat{\underline{\gamma}}_t)$ and where the elements of the design sequence play the role of the control variables. In this manner the design sequence controls the entire evolution of the error covariance matrix. Thus, in essence, the design problem that the experimenter is concerned with is as follows: find an ℓ_{jt}^*, which satisfies the imposed constraints (6) and (7), such that the estimates $\hat{\underline{\gamma}}_t^*$ of the "state" $\underline{\gamma}_t$ is best in some sense, where $\hat{\underline{\gamma}}_t^*$ is generated by the Kalman filter which corresponds to the ℓ_{jt}. The fact that we are seeking a design sequence ℓ_{jt}^* which is optimal, forces us to define precisely an index

[4] To keep the problem from becoming trivial, it is assumed that $c_t^{\nu}T > \Lambda_T$ where c_t^{ν} is the cost of the most expensive design point.

[5] For a discussion of a number of approaches in the measurement and valuation of information see, for example, [48].

of performance. A sensible design objective function can be built around $\mathbf{V}(\hat{\underline{\gamma}}_T)$. Suppose the researcher's goal is accurate estimation of a vector $\underline{\gamma}_T^o = \mathbf{P}\underline{\gamma}_T$ of linear combinations of the elements of $\underline{\gamma}_T$, where $\mathbf{P}$ is a known matrix of dimensions $p \times m$. The dimension p may be either larger or smaller than m. The estimate of $\underline{\gamma}_T^o$ and its covariance are $\hat{\underline{\gamma}}_T^o = \mathbf{P}\hat{\underline{\gamma}}_T$ and $\mathbf{V}(\hat{\gamma}_T^o) = \mathbf{P}\mathbf{V}(\hat{\underline{\gamma}}_T)\mathbf{P}'$, respectively. If the experimenter wishes to minimize the weighted sum of variances of the elements of $\hat{\underline{\gamma}}_T^o$, the objective function may be written as

$$P(\mathbf{u}) = \operatorname{tr}\left\{\mathbf{C}\left(\mathbf{V}\left[\hat{\underline{\gamma}}_T\right]\right)\right\}, \tag{8}$$

where $\operatorname{tr}(\cdot)$ is the trace operator, $\mathbf{C} = (\mathbf{P}'\mathbf{W}\mathbf{P})$ and $\mathbf{W}$ denotes the $p \times p$ diagonal matrix, whose diagonal elements $(W_1, \dots, W_p)$ indicate the policy importance to the experimenter (or the sponsor of the experiment) of the elements of $\underline{\gamma}_T^o = \underset{\sim}{P}\underline{\gamma}_T$.[6] Although, in this paper, we will focus on the trace criterion (L-optimality),[7] other suitable optimality criteria for optimal designs include the minimization of $\det(\mathbf{I}^{-1})$ (D-optimality), or of the maximum eigenvalue of $\mathbf{I}^{-1}$ (E-optimality)[8].

The system corresponding to model (1) can be written as

$$z_t = \left(\sum_{j=1}^{n} \ell_{jt}\mathbf{u}_{jt}\right)\underline{\gamma}_t + e_t. \tag{9}$$

It is clear that, since $\mathbf{u}_t = \left(\sum_{j=1}^{n} \ell_{jt}\mathbf{u}_{jt}\right)$, we can write the Kalman filter and its variance-covariance as follows:

$$\begin{aligned}
\hat{\underline{\gamma}}_t = &\left(\left[\underset{\sim}{\Gamma}_t \hat{\underline{\gamma}}'_{t-1}\right] + \left[\underset{\sim}{\Gamma}_t \mathbf{V}\left(\hat{\underline{\gamma}}_{t-1}\right)\underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right]\left[\sum_{j=1}^{n}\ell_{jt}\mathbf{u}_{jt}\right]'\right. \\
&\times \left[\sigma^2 + \sum_{j=1}^{n}\ell_{jt}\mathbf{u}_{jt}\left(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right]\underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right)\mathbf{u}_{jt}\right]^{-1} \\
&\left.\times \left[z_t - \left(\sum_{j=1}^{n}\ell_{jt}\mathbf{u}_{jt}\right)\left(\underset{\sim}{\Gamma}_t \hat{\underline{\gamma}}_{t-1}\right)\right]\right),
\end{aligned} \tag{10}$$

$$\begin{aligned}
\mathbf{V}(\hat{\underline{\gamma}}_t) = &\left(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right]\underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right) - \left(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right]\underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right) \\
&\times \left[\sum_{j=1}^{n}\ell_{jt}u'_j\left(\sum_{j=1}^{n}\ell_{jt}\mathbf{u}_{jt}\left[\underset{\sim}{\Gamma}_t \mathbf{V}\left(\hat{\underline{\gamma}}_{t-1}\right)\underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right]\mathbf{u}'_j + \sigma^2\right)^{-1}\mathbf{u}_{jt}\right] \\
&\times \left(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right]\underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right),
\end{aligned} \tag{11}$$

where use has been made of the fact that $\ell_{jt}^2 = \ell_{jt}$ and

$$\ell_{jt}\ell_{j't}\left(\mathbf{u}'_{jt}\left[\underset{\sim}{\Gamma}_t \mathbf{V}\left(\hat{\underline{\gamma}}_{t-1}\right)\underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right]\mathbf{u}_{j't}\right) = \ell_{jt}\ell_{j't}(\mathbf{u}'_{jt}\sigma^{-2}\mathbf{u}_{j't}) = 0,$$

since either ℓ_{jt} or $\ell_{j't}$ is zero, for all $j \neq j'$.

[6] Henceforth, for simplicity, we shall no longer indicate that $\hat{\underline{\gamma}}_T$ and $V(\hat{\underline{\gamma}}_T)$ depend on $\mathbf{u}$.

[7] It can be shown that the L-optimality criterion can provide an attractive appropriate measure of the value of information in the context of decision making under uncertainty, given an appropriate choice of the $\mathbf{C}$ matrix.

[8] The determinant and eigenvalue criteria are discussed, along with other standard (Bayesian) criteria, in [49,50].

Given the initial condition $\mathbf{V}(\underline{\gamma}_o) = \hat{\underset{\sim}{\Sigma}}_{\gamma_o}$, it can be shown by continuous substitution in (5a) that minimizing $\operatorname{tr}(\mathbf{C}\mathbf{V}[\underline{\hat{\gamma}}_T])$ with respect to $(\{\mathbf{u}_{jt} \text{ or } \ell_{jt}\}_{j=1}^{n};\ t = 1,2,\dots,T)$ is equivalent to maximizing Φ where

$$\begin{aligned}
\Phi = \operatorname{tr}\Bigg\{ \sum_{t=1}^{T} \underset{\sim}{\Xi}_t \Bigg[& \left(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\underline{\hat{\gamma}}_{t-1}\right] \underset{\sim}{\Gamma}_t' + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right) \\
& \times \left(\sum_{j=1}^{n} \ell_{jt}\mathbf{u}_{jt}' \left[\sum_{j=1}^{n} \ell_{jt}\mathbf{u}_{jt} \left(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\underline{\hat{\gamma}}_{t-1}\right] \underset{\sim}{\Gamma}_t' \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right)\mathbf{u}_{jt}' + \sigma^2 \right]^{-1} \mathbf{u}_{jt} \right) \\
& \times \left(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\underline{\hat{\gamma}}_{t-1}\right] \underset{\sim}{\Gamma}_t' + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\right) \Bigg] \Bigg\},
\end{aligned} \tag{12}$$

$$\underset{\sim}{\Xi} = \left(\left[\prod_{j=t}^{T} \underset{\sim}{\Gamma}_j \right]' \mathbf{C} \left[\prod_{j=t}^{T} \underset{\sim}{\Gamma}_j \right]' \right), \tag{13}$$

with respect to $(\{\mathbf{u}_{jt} \text{ or } \ell_{jt}\}_{j=1}^{n};\ t = 1,2,\dots,T)$.

The problem facing the investigator now can be stated as follows: for a given terminal time $T > 0$, determine the optimal sequence for the probing action $(\{\ell_{jt}\}_{j=1}^{n};\ t = 1,2,\dots,T)$ such that the criterion function given by (12), Φ, is maximized subject to: (i) (11); (ii) (6) and (7); (iii) $\mathbf{V}(\underline{\gamma}_o) = \hat{\underset{\sim}{\Sigma}}_{\gamma_o}$ (given positive definite matrix) and $\mathbf{V}(\underline{\hat{\gamma}}_T)$: unrestricted.[9]

3. OPTIMAL ESTIMATION STRATEGY FOR THE PROBING ACTION

We shall use the discrete maximum principle of Pontryagin (see, for example, [51]) to derive a set of necessary conditions for optimality.[10] Before we apply the maximum principle[11,12], however, it is necessary to transform the total budget constraint

$$\sum_{t=1}^{T} \left(\sum_{j=1}^{n} \ell_{jt} c_{jt} \right) = \Lambda_T$$

into a difference equation type constraint. To do this, we define a new state variable

$$\tilde{\omega}_{ot} = \sum_{k=1}^{t} \left(\sum_{j=1}^{n} \ell_{jk} c_{jk} \right).$$

It is apparent that $\tilde{\omega}_{ot}$ satisfies the first order difference equation

$$(\tilde{\omega}_{ot} - \tilde{\omega}_{ot-1}) = \sum_{j=1}^{n} \ell_{jt} c_{jt},$$

with $\tilde{\omega}_{oo} = 0$ (initial condition) and $\tilde{\omega}_{oT} = \Lambda_T$ (terminal condition).

---

[9]Note that this estimation design problem for the probing action is a discrete time optimal control problem, where the elements $\mathbf{V}(\underline{\hat{\gamma}}_t)_{hh'}$, $\{h, h' = 1, 2, \dots, (m \times m)\}$ of the variance-covariance matrix $\mathbf{V}(\underline{\hat{\gamma}}_t)$ play the role of the state variables of a dynamic system whose "equation of motion" is governed by the matrix variance-covariance difference Equation (11), ℓ_{jt}'s play the role of control variables, and the "cost functional" (objective function) depends on the values of the control and state variables $\ell_{jt}, \mathbf{V}(\underline{\hat{\gamma}}_t)_{hh'}$, for all t.

[10]The discrete maximum principle is essentially equivalent to the Kuhn-Tucker theorem (see, for example, [52]).

[11]For precise conditions under which the discrete maximum principle is derived, see [51,53].

[12]The discrete maximum principle yields, in general, a set of (local) necessary conditions for optimality.

We now define the real-valued function H, called the "Hamiltonian" which, using the constraints (6)–(7) and the properties of the trace function, can be written as follows:

$$
\begin{aligned}
H &= H(\mathbf{V}[\underline{\hat{\gamma}}_t], \underset{\sim}{\Omega}_t, \tilde{\omega}_{ot}, p_{ot}, \ell_{jt}, t) \\
&= \Bigg\{ \sum_{j=1}^{n} \ell_{jt} \Big[(p_{ot} c_{jt}) + \operatorname{tr}\Big(\Big[\mathbf{u}_{jt} [\underset{\sim}{\Gamma}_t \mathbf{V}(\underline{\hat{\gamma}}_{t-1}) \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big] \mathbf{u}_{jt} + \sigma^2 \Big)^{-1} \mathbf{u}_{jt} \Big] \\
&\quad \times \Big[\Big(\underset{\sim}{\Gamma}_t \mathbf{V}[\underline{\hat{\gamma}}_{t-1}] \underset{\sim}{\Gamma}_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big) \Big(\underset{\sim}{\Xi}_t - \underset{\sim}{\Omega}'_t \Big) \Big(\underset{\sim}{\Gamma}_t \mathbf{V}[\underline{\hat{\gamma}}_{t-1}] \underset{\sim}{\Gamma}_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big) \Big] \Big) \Big] \\
&\quad + \operatorname{tr}\Big(\Big\{ \Big[\underset{\sim}{\Gamma}_t \mathbf{V}\left(\underline{\hat{\gamma}}_{t-1}\right) \underset{\sim}{\Gamma}_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big] - \Big[\mathbf{V}\left(\underline{\hat{\gamma}}_{t-1}\right) \Big] \Big\} \underset{\sim}{\Omega}'_t \Big) \Bigg\},
\end{aligned} \tag{14}
$$

where $\{p_{ot};\ t = 1, 2, \ldots, T\}$ is the costate at time t corresponding to the state variable $\tilde{\omega}_{ot}$ and $\{\underset{\sim}{\Omega}_t;\ t = 1, 2, \ldots, T\}$ is the costate matrix (whose hh'^{th} element is the costate which corresponds to the $[\mathbf{V}(\underline{\hat{\gamma}}_t)]hh'$ state variable) corresponding to the variance-covariance matrix $\mathbf{V}(\underline{\hat{\gamma}}_t)$.

Assume that an optimal design sequence for the probing action exists. Let $\{\ell^0_{jt}\}^n_{j=1}$ denote the optimal design sequence; let $\{\mathbf{V}(\underline{\hat{\gamma}}_t)_0\}$ be the resultant variance-covariance matrix and let $\{\tilde{\omega}^0_{0t})$ be the resulting state variable. Then there exist costate variables $\{p^0_{0t}\}$ and $\{\tilde{\omega}^0_{hh'};\ h, h' = 1, 2, \ldots, (m \times m)\}$ such that the following conditions hold.

CONDITION 1. HAMILTONIAN MAXIMIZATION

The inequality

$$
\left(\sum_{j=1}^{n} \ell^0_{jt} \Delta^0_{jt} \right) \geq \left(\sum_{j=1}^{n} \ell_{jt} \Delta^0_{jt} \right), \tag{15}
$$

$$
\begin{aligned}
\Delta^0_{jt} = \Big\{ (p^0_{0t} c_{jt}) &+ \operatorname{tr}\Big(\Big[\mathbf{u}'_{jt} \Big(\mathbf{u}_{jt} \Big[\underset{\sim}{\Gamma}_t \mathbf{V}\left(\underline{\hat{\gamma}}_{t-1}\right)_0 \underset{\sim}{\Gamma}_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big] \mathbf{u}'_j + \sigma^2 \Big)^{-1} \\
&\times \mathbf{u}_{jt} \Big[\Big(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\underline{\hat{\gamma}}_{t-1}\right]_0 \underset{\sim}{\Gamma}_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big) \Big(\underset{\sim}{\Xi}_t - \underset{\sim}{\Omega}^{0'}_t \Big) \Big(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\underline{\hat{\gamma}}_{t-1}\right]_0 \underset{\sim}{\Gamma}_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big) \Big] \Big) \Big\},
\end{aligned} \tag{16}
$$

must hold for each $t = 1, 2, \ldots, T$ and for all $\ell_{jt} \in (0, 1)$, $\sum_{j=1}^{n} \ell_{jt} = 1$. In view of the above constraints on ℓ_{jt}, we have the following result:

$$
\ell^0_{jt} = \left\{ \begin{array}{ll} 1, & \text{if } [\Delta^0_{jt} \geq \Delta_{j't}] \text{ for all } j,\ j = 1, 2, \ldots, n;\ j \neq j' \\ 0, & \text{otherwise} \end{array} \right\}, \tag{17}
$$

where

$$
\begin{aligned}
\Delta_{j't} = (p^0_{0t} c_{j't}) &+ \operatorname{tr}\Big(\Big[\mathbf{u}'_{jt} \underset{\sim}{\Gamma}_t \mathbf{V}\left(\underline{\hat{\gamma}}_{t-1}\right)_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big] \mathbf{u}'_{j't} t + \sigma^2 \Big)^{-1} \\
&\times \mathbf{u}_{j't} \Big[\Big(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\underline{\hat{\gamma}}_{t-1}\right]_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big) \left(\underset{\sim}{\Xi}_t - \underset{\sim}{\Omega}^{0'}_t \right) \left(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\underline{\hat{\gamma}}_{t-1}\right]_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \right) \Big] \Big).
\end{aligned} \tag{18}
$$

CONDITION 2. CANONICAL EQUATIONS AND BOUNDARY CONDITIONS

$$
\begin{aligned}
\left(\mathbf{V}\left[\underline{\hat{\gamma}}_t\right]_0 - \mathbf{V}\left[\underline{\hat{\gamma}}_{t-1}\right]_0 \right) &= \left(\left. \frac{\partial H(\cdot)}{\partial \underset{\sim}{\Omega}_t} \right|_0 \right) \\
&= \Big\{ \Big(\Big[\underset{\sim}{\Gamma}_t \mathbf{V}\left(\underline{\hat{\gamma}}_{t-1}\right)_0 \underset{\sim}{\Gamma}_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big] - \Big[\mathbf{V}\left(\underline{\hat{\gamma}}_{t-1}\right)_0 \Big] \Big) \\
&\quad - \Big(\Big[\underset{\sim}{\Gamma}_t \mathbf{V}\left(\underline{\hat{\gamma}}_{t-1}\right)_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big] \Big) \\
&\quad \times \left[\sum_{j=1}^{n} \ell_{jt} \Big[\mathbf{u}_{jt} \Big(\mathbf{u}_{jt} \Big(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\underline{\hat{\gamma}}_{t-1}\right]_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big) \mathbf{u}'_{jt} + \sigma^2 \Big)^{-1} \mathbf{u}'_{jt} \Big] \right] \\
&\quad \times \Big[\underset{\sim}{\Gamma}_t \mathbf{V}\left(\underline{\hat{\gamma}}_{t-1}\right)_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t} \Big] \Big) \Big\},
\end{aligned} \tag{19}
$$

$$
(\tilde{\omega}^0_{0t} - \tilde{\omega}^0_{0t-1}) = \left(\sum_{j=1}^{n} \ell^0_{jt} c_{jt} \right), \tag{20}
$$

$$
\begin{aligned}
\left(\underset{\sim}{\Omega}_t^0 - \underset{\sim}{\Omega}_{t-1}^0\right) &= -\left(\frac{\partial(\cdot)}{\partial(\mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right])}\Bigg|_0\right) \\
&= -\Big\{\Big(\sum_{j=1}^{n} \ell_{jt}^0 \Big[\Big(\mathbf{u}'_{jt}\Big[\mathbf{u}_{jt}\Big(\underset{\sim}{\Gamma}_t V\left[\hat{\underline{\gamma}}_{t-1}\right]_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\Big)\mathbf{u}'_{jt} + \sigma^2\Big]^{-1} \\
&\quad \times \mathbf{u}_{jt}\Big[\underset{\sim}{\Gamma}'_t \mathbf{V}\left(\hat{\underline{\gamma}}_{t-1}\right)_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\Big]\underset{\sim}{\Gamma}'_t\Big[\underset{\sim}{\Xi}_t - \underset{\sim}{\Omega}_t^{0\prime}\Big]\underset{\sim}{\Gamma}_t\Big) \\
&\quad - \Big(\underset{\sim}{\Gamma}'_t\Big[\mathbf{u}_{jt}\Big(\underset{\sim}{\Gamma}_t \mathbf{V}\left[\hat{\underline{\gamma}}_{t-1}\right]_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\Big)\mathbf{u}'_{jt} + \sigma^2\Big]^{-1} \qquad (21)\\
&\quad \times \mathbf{u}_{jt}\Big[\underset{\sim}{\Gamma}_t \mathbf{V}(\hat{\underline{\gamma}}_{t-1})_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\Big]\Big[\underset{\sim}{\Xi}_t - \underset{\sim}{\Omega}_t^{0\prime}\Big]\Big[\underset{\sim}{\Gamma}_t \mathbf{V}(\hat{\underline{\gamma}}_{t-1})_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\Big] \\
&\quad \times \mathbf{u}'_{jt}\Big[\underset{\sim}{\Gamma}_t V[\hat{\underline{\gamma}}_{t-1}]_0 \underset{\sim}{\Gamma}'_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\Big)\mathbf{u}'_{jt} + \sigma^2\Big]^{-1} \mathbf{u}_{jt}\underset{\sim}{\Gamma}_t\Big) \\
&\quad + \Big(\Big[\underset{\sim}{\Xi}_t - \underset{\sim}{\Omega}_t^{0\prime}\Big]\Big[\underset{\sim}{\Gamma}_t \mathbf{V}(\hat{\underline{\gamma}}_{t-1})_0 \underset{\sim}{\Gamma}_t + \underset{\sim}{\Sigma}_{\mathbf{v}_t}\Big] \\
&\quad \times \Big[\mathbf{u}'_{jt}\Big[\underset{\sim}{\Gamma}_t \mathbf{V}(\hat{\underline{\gamma}}_{t-1})\underset{\sim}{\Gamma}'_t \underset{\sim}{\Sigma}_{\mathbf{v}_t}\Big]\mathbf{u}_{jt} + \sigma^2\Big)^{-1} \mathbf{u}_{jt}\Big]\Big[\underset{\sim}{\Gamma}'_t \underset{\sim}{\Gamma}_t\Big]\Big)\Big]\Big) \\
&\quad - \Big(\underset{\sim}{\Gamma}'_t \underset{\sim}{\Omega}_t^0 \underset{\sim}{\Gamma}_t - I\Big)\Big\},
\end{aligned}
$$

$$(p_{0t}^0 - p_{0t-1}^0) = 0. \tag{22}$$

$$\mathbf{V}(\underline{\gamma}_0) = \underset{\sim}{\hat{\Sigma}}_{\gamma_o} \text{ and } \omega_{00}^0 = 0 \text{ at the initial time } t = 0, \tag{23}$$

$$\underset{\sim}{\Omega}_T^0 = 0, \quad \underset{\sim}{\Omega}_{0T}^0 = \Lambda_T \text{ at the terminal time } t = T. \tag{24}$$

The above conditions can be used to determine, in each particular case, the optimal input signal $(\{\ell_{jt}^0;\ j = 1,2,\ldots,n\}_{t=1}^T)$ for the probing action.[13] This can be used in the estimation phase to quickly estimate the unknown parameters of the model up to a desirable accuracy, before the control policies are implemented, and complete learning will eventually take place during the control phase. Then the more precise estimetes are used in the control phase to determine the control law.[14]

4. CONCLUSIONS

It is known, in stochastic control theory, that the dual control optimizing a given functional of the system inputs and outputs must spend some energy in learning about the unknown parameters (and/or state) of the system under investigation (probing action). Some characteristics of such estimating parts of closed-loop control laws in the context of a regression model with time-varying parameters, have been investigated and an alternative proposal for a suboptimal dual controller

[13] A number of algorithms that use the necessary conditions of the maximum principle to obtain in an interactive manner numerical solutions to the optimal control problem that have been proposed in the literature (standard gradient techniques, for example) cannot be used in the context of our model because of the constraints on ℓ_{jt}. Other techniques, however, could be used to obtain numerical solutions for the optimal estimation control problem. The essence of one such technique can be outlined as follows: (a) an initial guess is made for the values of ℓ_{jt} (say ℓ_{jt}^* for all $j = 1,2,\ldots,n$; $t = 1,2,\ldots,T$); (b) the initial guess permits the solution for $\tilde{\omega}_{0t}$ and $\mathbf{V}(\hat{\underline{\gamma}}_t)$ forward in time (starting at the known condition ($\tilde{\omega}_{00} = 0$, $\mathbf{V}[\underline{\gamma}_t] = \hat{\Sigma}_{\gamma 0}$) as well as for $\underset{\sim}{\Omega}_t$ backward in time (using the known condition $\underset{\sim}{\Omega}_T = 0$); (c) the maximization condition (18) may now be used to determine new values for ℓ_{jt} (say ℓ_j^{**}); (d) steps two and three are repeated until a suitable criterion (based on the change in the objective function with successive iterations, perhaps) is satisfied.

[14] The overall control cost function resulting from the first (estimation) phase period will be rather high because the precise parameter criterion and cost function are generally contradictory. Hence, the longer the probing period is, the more precise the model parameters are, and at the same time, the worse is the control. Having more precise model parameters, however, we obtain a better control law (lower value of the cost function) during the second (control) phase.

is introduced in this paper. It is based on the combination on the experimental design for parameter estimation and the passive adaptive control law. The probing action aims to decrease the uncertainty of the parameter estimates measured by the weighted sum of variances of the elements of the estimated parameters at time T. Then the more precise estimates are used in the control action. The experimental design used for the probing action and the sequential estimation method based on Kalman filtering combined together in the proposed procedure are consistent from the computational point of view.

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